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Simple Lie Algebras with Classical Reductive Null Component

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INTRODUCTION

In [2], Kac proposed a program to classify the simple finite-dimensional Lie algebras over an algebraically closed field of characteristic greater than three. The first (and apparently most difficult) step in this program is finding an appropriate maximal subalgebra L_0 of a given Lie algebra L: it is required that L_0 determine a filtration $L = L_{-k} \supseteq \cdots \supseteq L_0 \supseteq \cdots$ in which L_0/L_1 is the direct sum of simple classical algebras and a possibly nontrivial center. Candidates for L_0 have been proposed by Kostrikin [4] and Wilson [12] and work is progressing in that area. The next step in the program is to determine the associated graded algebra $\operatorname{gr} L$ when L_0/L_1 is known. The final step is to find the filtered Lie algebras with which a particular graded Lie algebra is associated.

Wilson [11], Kostrikin and Šafarevič [6], and Kac [2, 3] have made significant progress in carrying out the last two steps. In connection with the next-to-last step, Kac [2, Theorem 3] proves the following theorem: Let $L = \oplus G_i$ be a finite-dimensional nontrivial transitive graded Lie algebra over an algebraically closed field of characteristic $p > 3$ satisfying the following conditions:

(a) The subalgebra G_0 is isomorphic to the direct sum of its center and some simple Lie algebras of classical type.

(b) The G_0-module G_{-1} is a p-representation.

Then G is isomorphic either to a Lie algebra of Cartan type or to a Lie algebra of classical type with one of the standard gradations, or to a Lie algebra of type A^z_{kp-1} (see the definition following Proposition 1.5 of [2]). The present author has shown that if it is only assumed that $L = G_{-1} \oplus G_0 \oplus \cdots \oplus G_k$ is a simple Lie algebra over an algebraically closed field of characteristic $p > 3$ such that condition (a) above holds, then G is isomorphic either to a Lie algebra of Cartan type, or to a Lie algebra of classical type with one of the standard gradations, or to a Lie algebra of type A^z_{kp-1}.

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In [5], Kostrikin solved a similar problem: he determined the simple graded Lie algebras whose zero component is isomorphic to W_1 and whose -2 component is zero. More recently, Kuznecov [7] showed that there are no simple graded Lie algebras whose null component is solvable.

0. Notation

Throughout this paper, L will denote a simple, graded Lie algebra over an algebraically closed field of characteristic $p > 3$. We will assume that L has the particular form

$$L = G_{-1} \oplus G_0 \oplus \cdots \oplus G_k, \tag{1}$$

where $G_k \neq \{0\}$, and where, as usual, the G_i are subspaces such that

$$[G_i, G_j] \subseteq G_{i+j} \qquad \text{whenever} \quad -1 \leqslant i+j \leqslant k, \tag{2}$$

and

$$[G_i, G_j] = \{0\}, \qquad \text{otherwise.} \tag{3}$$

Since $[G_0, G_0] \subseteq G_0$, G_0 is a subalgebra. We assume that G_0 is isomorphic to the direct sum of its center and some simple Lie algebras of classical type:

$$G_0 = Z \oplus G_{0,1} \oplus G_{0,2} \oplus \cdots \oplus G_{0,b} = Z \oplus [G_0, G_0], \tag{4}$$

where Z may be zero, and $b \geqslant 1$. Let

$$\alpha_1, \ldots, \alpha_\rho \tag{5}$$

be a system of simple roots of $[G_0, G_0]$. Let e_i, f_i, and h_i be vectors in the spaces associated with the roots α_i, $-\alpha_i$, and zero, respectively, such that

$$[e_i, f_i] = h_i, \qquad [e_i, h_i] = 2e_i, \qquad \text{and} \qquad [f_i, h_i] = -2f_i; \tag{6}$$

also,

$$[e_{i_1}, f_{i_2}] = 0 \tag{7}$$

if i_1 differs from i_2. In view of (2), we have that $[G_0, G_j] \subseteq G_j$ for each j. The Jacobi identity in L shows that for $g_0, g_0' \in G_0$ and $g_j \in G_j$, we have

$$[g_j, [g_0, g_0']] = [[g_j, g_0], g_0'] - [[g_j, g_0'], g_0];$$

hence, if we take ad_{G_j} to mean the restriction to G_j of the adjoint representation of L in itself, then $g_0 \rightarrow \mathrm{ad}_{G_j} g_0$ gives a representation of G_0 in G_j. When we speak of a weight vector of G_j, we will mean a weight vector with respect to this representation of G_0 in G_j. (When $j = 0$, we consider the terms "weight vector" and "root vector" to be synonymous.) For each t, $1 \leqslant t \leqslant b$, we fix a highest-root vector r_t of $G_{0,t}$; i.e., r_t is a root vector of $G_{0,t}$ which is annihilated by all e_i.

The particular nonzero scalar multiple of a vector will often be of little interest in an argument. We have therefore found it convenient to use the symbol "$\doteq$," which will mean "is a nonzero scalar multiple of."

1. The Main Result

Our aim is to prove the following

Theorem. *Let $L = G_{-1} \oplus G_0 \oplus \cdots \oplus G_k$ be a simple, finite-dimensional, graded Lie algebra over an algebraically closed field of characteristic $p > 3$, and suppose that G_0 is isomorphic to the direct sum of its (possibly trivial) center and some simple Lie algebras of classical type. Then G is isomorphic either to a Lie algebra of Cartan type or to a Lie algebra of classical type with one of the standard gradations, or to a Lie algebra of type A^z_{kp-1}.*

Remark. This theorem generalizes the theorem in Chapter II of [6], where the dimension of G_{-1} is assumed to be less than $p - 1$ and the representation of G_0 in G_{-1} induced by the bracket operation of L is assumed to be an irreducible p-representation; also, as noted in the Introduction, this theorem generalizes Theorem 3 of [2] (for L as in (1)).

2. Simple Properties of Simple Lie Algebras with Gradation

We begin by showing that G_{-1} and G_k are nontrivial, irreducible G_0-modules, and that $[G_{-1}, x]$ is nonzero for any (nonzero) element x of L which is not contained in G_{-1}. Graded Lie algebras satisfying this latter condition are often called "transitive," while those in which G_{-1} is an irreducible G_0-module are usually referred to as "irreducible" (see, for example, Section 1 of [2].)

Lemma 1. *If i is greater than -1, then no nonzero element of G_i is annihilated by G_{-1}. Furthermore, G_{-1} is a nontrivial, irreducible G_0-module.*

Proof. First of all, we note that if x is a nonzero element of L which is annihilated by G_{-1}, then

$$I = \langle x \rangle + \sum_{j_1,\ldots,j_n \geqslant 0} [\cdots [x, G_{j_1}],\ldots, G_{j_n}] \tag{8}$$

is a nonzero ideal of L. It suffices to show that I is stable under ad G_{-1} ; one easily shows $[[\cdots [x, G_{j_1}],..., G_{j_n}], G_{-1}] \subseteq I$ by the Jacobi identity and induction on n. If $x \in G_i \setminus \{0\}$ for some $i > -1$ and $[G_{-1}, x] = \{0\}$, then $I \subseteq G_i \oplus \cdots \oplus G_k \neq L$, so that I would be a proper ideal of the simple algebra L. This contradiction proves the first assertion of the lemma.

Now let M be a nonzero G_0-submodule of G_{-1}, and let x be a nonzero element of M. By (3), $[G_{-1}, x] = \{0\}$. Let I be defined as in (8). Then $I \neq \{0\}$, so $I = L$, since L is simple. Because

$$I \cap G_{-1} = \langle x \rangle + \sum_{n>0} [\cdots [x, \underbrace{G_0],..., G_0]}_{n \text{ copies}},$$

it is clear that $M \supseteq I \cap G_{-1}$. Therefore, we have

$$G_{-1} \supseteq M \supseteq I \cap G_{-1} = L \cap G_{-1} = G_{-1},$$

so that $M = G_{-1}$. Since M was any nonzero G_0-submodule of G_{-1}, it follows that G_{-1} is an irreducible G_0-module.

If G_{-1} were a trivial G_0-module (i.e., if $[G_{-1}, G_0] = \{0\}$), then $[L, L] = \sum_{-1 \leqslant i,j \leqslant k} [G_i, G_j] \subseteq G_0 \oplus \cdots \oplus G_k \neq L$, so that $G_0 \oplus \cdots \oplus G_k$ would be a proper ideal of L. This contradiction shows that G_{-1} must be a nontrivial G_0-module, and completes the proof of the lemma.

LEMMA 2. *G_k is an irreducible G_0-module. Furthermore, for $0 \leqslant i \leqslant k$, we have that $[G_{-1}, G_i] = G_{i-1}$.*

Proof. For any nonzero x in G_k,

$$J = \langle x \rangle + \sum_{\substack{i \geqslant 0 \\ j \geqslant 0 \\ i+j \geqslant 1}} [\cdots [[\cdots [x, \underbrace{G_0],..., G_0]}_{i \text{ copies}}, \underbrace{G_{-1}],..., G_{-1}]}_{j \text{ copies}}$$

is a nonzero ideal of L, since by the Jacobi identity and induction on j, each summand in the sum on the right above is stable under G_n, $n \geqslant 1$. Since L is simple, $J = L$. Because

$$J \cap G_k = \langle x \rangle + \sum_{i>0} [\cdots [x, \underbrace{G_0],..., G_0]}_{i \text{ copies}},$$

it follows, as in the proof of Lemma 1, that for M a nonzero G_0-submodule of G_k and x a nonzero element of M, we have for J defined as above in terms of this x that

$$G_k \supseteq M \supseteq J \cap G_k = L \cap G_k = G_k.$$

Therefore, $M = G_k$, and G_k is seen to be an irreducible G_0-module.

Finally, for any j such that $0 \leqslant j \leqslant k+1$, we have

$$G_{k-j} = G_{k-j} \cap L = G_{k-j} \cap J = \sum_{i \geqslant 0} [\cdots [[\cdots [x, \underbrace{G_0], \ldots, G_0]}_{i \text{ copies}}, \underbrace{G_{-1}], \ldots, G_{-1}]}_{j \text{ copies}}.$$

In particular,

$$G_{k-j} = [G_{k-(j-1)}, G_{-1}],$$

which is just the second assertion of the lemma.

LEMMA 3. *If for some m and n such that $0 \leqslant m, n \leqslant k$ we have $[G_{-1}, G_m] = G_{m-1}$, $[G_m, G_n] = \{0\}$, and $[G_{-1}, G_n] = G_{n-1}$, then $[G_{m-1}, G_n] = [G_m, G_{n-1}]$.*

Proof. In view of the Jacobi identity, we have

$$[[G_m, G_{-1}], G_n] \subseteq [[G_m, G_n], G_{-1}] + [G_m, [G_{-1}, G_n]].$$

By our assumptions, the subspace on the left is equal to $[G_{m-1}, G_n]$, the first summand on the right is equal to zero, and the second summand on the right is equal to $[G_m, G_{n-1}]$; hence, $[G_{m-1}, G_n] \subseteq [G_m, G_{n-1}]$. Inclusion in the other direction holds by symmetry, and the lemma follows.

LEMMA 4. *G_k is a nontrivial G_0-module.*

Proof. By (3), we have

$$[G_i, G_j] = \{0\} \qquad \text{for} \quad i + j = k + 1,$$

so, in view of the second assertion of Lemma 2, we have by repeated application of Lemma 3 that

$$[G_0, G_k] = [G_1, G_{k-1}] = \cdots = [G_i, G_{k-i}] = \cdots = [G_k, G_0].$$

Hence, if G_k were a trivial G_0-module, then all of the above subspaces would be zero. In view of (1), we would then have

$$[L, L] = \sum_{-1 \leqslant i, j \leqslant k} [G_i, G_j] \subseteq G_{-1} \oplus \cdots \oplus G_{k-1},$$

and the sum on the right above would be a proper ideal of L, contradicting our assumption that L is simple. Q.E.D.

LEMMA 5. *For each j such that $-1 \leqslant j \leqslant k$, we have $G_j \neq \{0\}$.*

Proof. We assumed at the outset (1) that $G_k \neq \{0\}$. If $G_j = \{0\}$ for some j such that $-1 \leqslant j < k$, then $\oplus_{n=j+1}^{k} G_n$ would be an ideal of L, contradicting the simplicity of L.

3. Highest-Weight Vectors in G_{-1}

If the representation of $[G_0, G_0]$ in G_{-1} is a p-representation, then one can conclude that G_{-1} contains a highest-weight vector for this representation. This section is devoted to showing that even when we do not assume that the representation is restricted, a highest-weight vector must nonetheless exist.

Lemma 6. *For each j such that $-1 \leqslant j \leqslant k$, the subspace G_j contains a nonzero weight vector for the representation of G_0 in G_j.*

Proof. Since we have assumed that G_j is a finite-dimensional vector space over an algebraically closed field, this lemma follows from Lemma 5 and the commutativity of the Cartan subalgebra of G_0.

Lemma 7. *For $-1 \leqslant i \leqslant k-1$, G_i is spanned by weight vectors of the form $[u, w]$, where u is a weight vector of G_{-1}, and w is a weight vector of G_{i+1}. G_k is spanned by weight vectors, also.*

Proof. By Lemma 6, G_{-1} contains a nonzero weight vector x for the representation of G_0 in G_{-1}. According to Lemma 1, G_{-1} is irreducible as a G_0-module, so G_{-1} must coincide with the G_0-submodule S of G_{-1} which is generated by x. Since G_0 is spanned by root vectors and the bracket of a root vector with a weight vector is a weight vector (or zero), it follows that G_{-1} $(= S)$ is spanned by weight vectors. Similarly, by Lemmas 6 and 2, G_k is spanned by weight vectors. Since the bracket of two weight vectors is again a weight vector (or zero), the first assertion of Lemma 7 now follows from the second assertion of Lemma 2 and induction. Q.E.D.

Lemma 8. *G_{-1} contains a weight vector v^- which is annihilated by all e_i.*

Proof. Fix t. Then $r_t = \sum_{j=1}^{m} [u_j, w_j]$ by Lemma 7, where u_j is a weight vector of G_{-1} and w_j is a weight vector of G_1. Since the root-space decomposition of $G_{0,t}$ is direct (see, for example, Sections I.5 and I.6 of [8]), and since the root space corresponding to the highest root is one dimensional (Lemma II.3.2 of [8]), we can assume that for every j, $[u_j, w_j]$ is either zero or a nonzero scalar multiple of r_t. Let j_0 be an integer such that $[u_{j_0}, w_{j_0}] \neq 0$. Then we set $v = u_{j_0}$ and $c = w_{j_0}$, and we have

$$[v, c] \doteq r_t .$$

By Lemma 1, $[G_{-1}, r_t] \neq \{0\}$; hence, there exists a weight vector $y \in G_{-1}$ such that $[r_t, y] \neq 0$. Suppose there were an infinite sequence of simple-root vectors $e_{s_1}, e_{s_2}, \ldots$ of G_0 such that

$$[\cdots [[r_t, y], e_{s_1}], \ldots, e_{s_n}] = [r_t, [\cdots [y, e_{s_1}], \ldots, e_{s_n}]] \tag{9}$$

were unequal to zero for all n. Since $r_n \doteq [v, c]$ and G_{-1} is Abelian, this would imply that

$$[[v, c], [\cdots [y, e_{s_1}], \ldots, e_{s_n}]] = [v, [c, [\cdots [y, e_{s_1}], \ldots, e_{s_n}]]]$$

were unequal to zero, so that

$$[c, [\cdots [y, e_{s_1}], \ldots, e_{s_n}]] \neq 0$$

for all n. But these are root vectors of G_0, so we would have an infinite sequence of roots of G_0, each obtained from the previous root by the addition of a simple (positive) root. This situation is known to be impossible when G_0 is classical (for the proof for prime characteristic, see [8, p. 31]). Hence, no such infinite sequence of nonzero vectors as (9) can exist; i.e., any sequence (9) must terminate in a vector which is annihilated by every simple-root vector e_i. We take v^+ to be such a terminal vector. Q.E.D.

4. Proof of the Main Result

The present author's original proof of the main result (for $k > 1$ and $G_0 = G_{0,1}$) proceeded along the following lines: if the difference in the weights of the highest- and lowest-weight vectors of G_{-1} were greater than the highest root of G_0, then the difference between the highest and lowest roots of G_0 would be greater than twice the highest root; i.e., we arrive at a contradiction. Using the fact that

$$v^+ \doteq [v^-, r_1], \tag{10}$$

where v^- is a lowest-weight vector of G_{-1}, one has that $\mathrm{ad}_{G_{-1}} r_1$ is a rank one transformation of G_{-1} and $(\mathrm{ad}_{G_0} r_1)^2 \neq 0$, so that Theorem 1 of [9] implies the result. Also, one can use (10) to show that G_0 must be A_n or C_n and the representation Γ of G_0 on G_{-1} must be the natural representation of G_0 on G_{-1}. Then Theorem 2 of Chapter III of [6] implies the result. Finally, (10) can be used to show that Γ is restricted, and one can then use Theorem 3 of [2] to get the result. The present author also proved the main result for $k = 1$ and G_0 simple; the referee pointed out that this case had already been dealt with in [2, 9].

The author wishes to thank the referee for suggesting the shorter reduction to Theorem 3 of [2] below. Also, thanks are due to Professor George B. Seligman for his help in generalizing the result and his formulation of the referee's ideas.

Proof of Theorem (Referee; Seligman). Since the choice of a root system (5) for $[G_0, G_0]$ was arbitrary, Lemma 8 shows that for every classical Cartan decomposition of $[G_0, G_0]$ with associated ordering of the roots, there is a nonzero weight vector v^+ in G_{-1} that is annihilated by all root vectors belonging to positive roots.

If e_i is a positive-root vector, then $(\mathrm{ad}_{G_0} e_i)^p = 0$. It follows that $e_i{}^p$ is in the center of the universal enveloping algebra $U(G_0)$, so acts as a scalar on G_{-1} by Schur's lemma. But e_i annihilates v^+, so $e_i{}^p$ does, too; the scalar must therefore be zero, and $e_i{}^p$ annihilates G_{-1}.

Now one can use the Weisfeiler–Kac linear function $l(x) = (\chi_{G_{-1}}(x^p - x^{[p]}))^{1/p}$ (see [10]): this annihilates all root vectors by the above remarks, based on Lemma 8, and $[G_0, G_0]$ is spanned by root vectors relative to classical Cartan decompositions. (For this, fix a classical Cartan decomposition. Two applications of the above give (i) $l(x) = 0$ if x is a positive-root vector; (ii) $l(x) = 0$ if x is a negative-root vector. It remains only to show that l vanishes on the Cartan subalgebra; but each of the automorphisms $\exp(\mathrm{ad}_{G_0} e_i)$ maps classical Cartan decompositions to classical Cartan decompositions. Thus,

$$l(f_i^{\exp(\mathrm{ad}_{G_0} e_i)}) = 0;$$

but

$$f_i^{\exp(\mathrm{ad}_{G_0} e_i)} = f_i + h_i + k_i e_i ,$$

where k_i is an element of the base field. It follows from (i), (ii), and the above that $l(h_i) = 0$.) Thus, $x^p - x^{[p]}$ acts as the scalar 0 for each $x \in [G_0, G_0]$ so G_{-1} is a p-module of $[G_0, G_0]$.

Now let y be any nonzero element of Z. Then by Schur's lemma, $\mathrm{ad}_{G_{-1}} y = \lambda I$, where I is the identity transformation of G_{-1} and λ is an element of the base field. By the first assertion of Lemma 1, $\lambda \neq 0$. Similarly, if y' is another element of Z, then $\mathrm{ad}_{G_{-1}} y' = \mu I$, where μ is an element of the base field. Then $\mathrm{ad}_{G_{-1}}(y' - (\mu/\lambda)y) = 0$, so, again by Lemma 1, $y' - (\mu/\lambda)y = 0$; i.e., $Z = \langle y \rangle$.

Let $z = -(1/\lambda)y$. Then $Z = \langle z \rangle$, and $\mathrm{ad}_{G_{-1}} z = -I$. We make Z into a p-algebra by defining $z^{[p]} = z$. Then

$$(\mathrm{ad}_{G_{-1}} z)^p = (-I)^p = -I = \mathrm{ad}_{G_{-1}} z = \mathrm{ad}_{G_{-1}} z^{[p]},$$

so G_{-1} is a p-module of Z, also.

The restriction of the representation Γ of G_0 in G_{-1} to each of the direct summands (4) of G_0 is a restricted representation. It follows that Γ is restricted. Our main result now follows from Theorem 3 of [2].

Appendix: Notation

L	a simple, graded Lie algebra over an algebraically closed field of characteristic $p > 3$
G_i	ith subspace in the gradation of L
z	vector spanning the center of G_0
$G_{0,t}$	tth simple Lie algebra summand of G_0
k	highest index of a gradation subspace of L
b	number of simple Lie algebra summands of G_0
α_i	ith simple root in an arbitrary ordering of simple roots of G_0
e_i	vector spanning the space associated with the root α_i
f_i	vector spanning the space associated with the root $-\alpha_i$
h_i	vector equal to the bracket of e_i with f_i
g_i	arbitrary vector in G_i
g_i'	arbitrary vector in G_i
$\doteq$	symbol meaning "is a nonzero scalar multiple of"
r_t	root vector of $G_{0,t}$ which is annihilated by e_i , $i = 1,..., \rho$
v^+	weight vector of G_{-1} which is annihilated by e_i , $i = 1,..., \rho$
$\chi_{G_{-1}}$	homomorphism from the center of the universal enveloping algebra of G_0 to the base field defined in [10]
Γ	representation of G_0 in G_{-1} induced by the bracket operation in L
ρ	number of simple roots of G_0

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